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Symplectic Orthogonality Spaces

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Let E be a finite dimensional vector space over the Galois field GF(2). Let $\operatorname{lin}(E)$ denote the set of one-dimensional subspaces of E. Let Φ be the symplectic inner product on E. Consider the elements of $\operatorname{lin}(E)$ as vertices of a graph, two vertices being connected exactly when they are distinct and orthogonal with respect to Φ. This graph is characterized abstractly.

1. INTRODUCTION: THE GENERAL PROBLEM

In the Foulis-Randall approach to the operational logic of an empirical science (see [3]), the notion of a sample space is generalized from the classical Kolmogorov idea to be a set X together with a relation $\perp$ which is assumed to be symmetric and irreflexive on X. The pair $(X, \perp)$ is referred to as an *orthogonality space.* Whereas the Kolmogorov approach produces only classical Boolean logics, the generalized statistics of Foulis and Randall produce more general logics, including the non-Boolean orthomodular structures that underly conventional non-relativistic quantum mechanics. Here, Xis the set of one-dimensional subspaces, the lines, of a complex Hilbert space and $\perp$ is the orthogonality relation among the lines given by the inner product. Foulis has queried whether this orthogonality space can be characterized by abstractly given internal properties of the space alone.

In [2], we studied algebraic generalizations of Hilbert spaces, namely, quadratic spaces, which still produce orthomodular structures. A *quadratic space* is a triple (k, E, Φ) where k is a division ring, E is a k-vector space, and Φ is a non-degenerate orthosymmetric sesquilinear form on E. We can consider the orthogonality space of lines of E, $\operatorname{lin}(E)$ with the usual orthogonality given by the form Φ, $\perp(\Phi)$, and ask if this orthogonality space can be characterized. Though this problem appears quite difficult, we present below a solution for a very special case. This is where k is the

Galois field of two elements and Φ is a symplectic form, that is, $\Phi(x, x) = 0$ for all x (see [1]). The theorem is as follows:

THEOREM. ***Let*** $(X, \perp)$ ***be a symplectic orthogonality space. Then there is a symplectic quadratic space*** $(\mathrm{GF}(2), E, \Phi)$ ***such that*** $(X, \perp)$ ***is isomorphic to*** $(\mathrm{lin}(E), \perp(\Phi))$.

For definitions, the reader is invited to continue on. It should be noted that, only with slight modification of point of view, this theorem can be considered as a result in graph theory.

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2. PROOF OF THE THEOREM

The proof of the theorem will be accomplished by a sequence of lemmas. First some notation. If X is a set, X will denote the cardinality of X. The union of disjoint sets will be symbolized $\dot\cup$.

An ***orthogonality space*** is a pair $(X, \perp)$ where X is a non-empty set and $\perp$ is a symmetric irreflexive relation on X. For x in X, define $x^\perp = \{y \in X \;\; x \perp y\}$ (the ***perp*** of x). Also define $\mathrm{co}(x^\perp) = \{y \in X \;\; x \not\perp y\}\backslash\{x\}$ (the ***co-perp*** of x). Clearly each x in X yields a decomposition of the space X into three mutually disjoint pieces: $X = \{x\} \dot\cup x^\perp \dot\cup \mathrm{co}(x^\perp)$. We call x the ***pivot*** of this decomposition.

Three distinct points (x, y, z) are said to form an ***odd triple*** iff for all v in $X\backslash\{x, y, z\}$ we have $v^\perp \cap \{x, y, z\}|$ is odd (and hence equal to 1 or 3).

LEMMA 1. ***Let (x, y, z) be an odd triple. Then***

(1) $\mathrm{co}(x^\perp)\backslash\{y, z\} = (y^\perp \cap [\mathrm{co}(x^\perp)\backslash\{y, z\}]) \dot\cup (z^\perp \cap [\mathrm{co}(x^\perp)\backslash\{y, z\}])$,

(2) $(x^\perp \cap y^\perp)\backslash\{z\} = (x^\perp \cap z^\perp)\backslash\{y\}$,

(3) $y^\perp$ n $z^\perp \cap \mathrm{co}(x^\perp) = \square$.

Proof. (1) Let $w \in \mathrm{co}(x^\perp)\backslash\{y, z\}$. Then $w \neq x, y, z$, and $w \not\perp x$. Thus w is orthogonal to exactly one of y or z. The other inclusion is clear.

(2) Let w ∈ ($x^\perp$ ∩ y'-)\(z). Then $w \perp x$, $w \perp y$ and so w is different from x, y, z. Hence $w \perp z$ so $w \in (x^\perp \cap z^\perp)\backslash\{y\}$.

(3) is clear from definitions.

A triple ***(a, b, c)*** is a ***fundamental triple*** iff (1) ***(a, b,*** c} is an orthogonal set, that is, $a \perp b$, $b \perp c$, and $c \perp a$, and (2) ***(a, b,*** c) is an odd triple.

LEMMA 2. *Let (a, b, c) be a fundamental triple. Then*

(1) $\mathrm{co}(a^\perp) = (b^\perp\ n\ \mathrm{co}(a^\perp)) \dot\cup (c^\perp\ n\ \mathrm{co}(a^\perp))$,

(2) $(a^\perp\ n\ b^\perp)\backslash\{c\} = (a^\perp\ n\ c^\perp)\backslash\{b\}$,

(3) $\{b\} \dot\cup (b^\perp\ n\ a^\perp) = \{c\} \dot\cup (c^\perp\ n\ a^\perp)$,

(4) $b^\perp\ n\ c^\perp\ n\ co(d) = \square$.

Proof. (1) Note that $\mathrm{co}(a^\perp) = \mathrm{co}(a^\perp)\backslash\{b, c\}$ and use Lemma l(1).

(2) is immediate from Lemma l(2).

(3) Let $w \in \{b\} \dot\cup (b^\perp\ n\ a^\perp)$. If $w = b$, then $w \in c^\perp\ n\ a^\perp$ so $w \in \{c\} \dot\cup (c^\perp\ n\ a^\perp)$. Suppose $w \neq b$. Then $w \in b^\perp\ n\ a^\perp$. ***Now*** if w = c we have $w \in \{c\} \dot\cup (c^\perp\ n\ a^\perp)$, so suppose $w \neq c$. Then $w \neq a$, ***b,*** c and w is orthogonal to a and ***b.*** Thus $w \perp c$, that, is $w \in c^\perp\ n\ a^\perp$. The other inclusion is obtained symmetrically.

LEMMA 3. *Let (a, b, c) and (a, b, c') be fundamental triples. Then*

(1) $c^\perp\ n\ \mathrm{co}(a^\perp) = c'^\perp\ n\ \mathrm{co}(a^\perp)$,

(2) $\{c\} \dot\cup (c^\perp\ n\ a^\perp) = \{c'\} \dot\cup (c'^\perp\ n\ a^\perp)$,

(3) ***either*** $c = c'$ ***or*** $c \perp c'$,

(4) $\{c\} \dot\cup c^\perp = \{c'\} \dot\cup c'^\perp$,

(5) $c^\perp\backslash\{c'\} = c'^\perp\backslash\{c\}$,

(6) $\mathrm{co}(c^\perp) = \mathrm{co}(c'^\perp)$.

Proof. (1) By Lemma 2(1), $c^\perp\ n\ \mathrm{co}(d) = \mathrm{co}(a^\perp)\backslash(b^\perp\ n\ \mathrm{co}(a^\perp)) = c'^\perp\ n\ \mathrm{co}(a^\perp)$.

(2) By Lemma 2(3),

$$\{c\} \dot\cup (c^\perp\ n\ a^\perp) = \{b\} \dot\cup (b^\perp\ n\ a^\perp) = \{c'\} \dot\cup (c'^\perp\ n\ a^\perp).$$

(3) is clear from (2).

(4) Since $c^\perp = \{a\} \dot\cup (c^\perp\ n\ a^\perp) \dot\cup (c^\perp\ n\ \mathrm{co}(a^\perp))$ we have

$$\begin{aligned}\{c\} \dot\cup c^\perp &= \{a\} \dot\cup \{c\} \dot\cup (c^\perp\ n\ a^\perp) \dot\cup (c^\perp\ n\ \mathrm{co}(a^\perp))\\ &= \{a\} \dot\cup \{c'\} \dot\cup (c'^\perp\ n\ a^\perp) \dot\cup (c'^\perp\ n\ \mathrm{co}(a^\perp))\\ &= \{c'\} \dot\cup c'^\perp.\end{aligned}$$

(5) is clear by (4).

(6) $\mathrm{co}(c^\perp) = X\backslash(\{c\} \dot\cup c') = X\backslash(\{c'\} \dot\cup c'^\perp) = \mathrm{co}(c'^\perp)$.

A triple ***(a, p, q)*** is a ***fundamental co-triple*** iff (1) they are distinct and $a \not\perp p$, $p \not\perp q$, and $q \not\perp a$, and (2) (a, ***p, q)*** is an odd triple.

LEMMA **4.** ***Let (a, p, q) be a fundamental co-triple. Then***

(1) $\mathrm{co}(a^\perp)\backslash\{p, q\} = (p^\perp \cap \mathrm{co}(a^\perp)) \cup (q^\perp \cap \mathrm{co}(a^\perp))$,

(2) $a^\perp \cap p^\perp = a^\perp \cap q^\perp$,

(3) $p^\perp \cap q^\perp \cap \mathrm{co}(a^\perp) = \square$,

(4) $q^\perp = (q^\perp \cap \mathrm{co}(a^\perp)) \cup (q^\perp \cap a^\perp)$,

(5) $\mathrm{co}(p^\perp) \cap a^\perp = \mathrm{co}(q^\perp) \cap a^\perp$.

Proof. (1) Note that

$$p^\perp \cap \mathrm{co}(a^\perp) = p^\perp \cap \mathrm{co}(a^\perp)\backslash\{p, q\}$$

and

$$q^\perp \cap \mathrm{co}(a^\perp) = q^\perp \cap \mathrm{co}(a^\perp)\backslash\{p, q\}$$

and ***use*** Lemma l(1).

(2) Let $w \in a^\perp \cap p^\perp$. Then $w \perp a$ and p so $w \neq q$ and so $w \neq a, p, q$. Thus $w \perp q$ so $w \in a^\perp \cap q^\perp$. The other inclusion is obtained symmetrically.

(3) Use Lemma l(3).

(4) Note that $q^\perp = q^\perp \cap X = (q^\perp \cap \{a\}) \cup (q^\perp \cap a^\perp) \cup (q^\perp \cap \mathrm{co}(a^\perp))$ and $q^\perp \cap \{a\} = \square$.

(5) is clear.

LEMMA 5. ***Let (a, p, q) and (a, p, q') be fundamental co-triples. Then***

(1) $a^\perp \cap q^\perp = a^\perp \cap q'^\perp$,

(2) either $q = q'$ ***or*** $q \perp q'$,

(3) $q^\perp\backslash\{q'\} = q'^\perp\backslash\{q\}$,

(4) $(q^\perp \cap \mathrm{co}(a^\perp))\backslash\{q'\} = (q'^\perp \cap \mathrm{co}(a^\perp))\backslash\{q\}$,

(5) $\{q\} \cup q^\perp = q'^\perp \cup \{q'\}$,

(6) $\mathrm{co}(q^\perp) = \mathrm{co}(q'^\perp)$.

Proof. (1) is immediate from Lemma 4(2).

(2) Suppose $q \neq q'$. Then

$$q' \in \mathrm{co}(a^\perp)\backslash\{p, q\} = (p^\perp \cap \mathrm{co}(a^\perp)) \cup (q^\perp \cap \mathrm{co}(a^\perp)).$$

But $q' \not\perp p$ so $q' \in q^\perp \cap \mathrm{co}(a^\perp)$ whence $q' \perp q$.

(3) Let $w \in q^\perp\backslash\{q'\}$. Then $w \perp q$ and $w \neq q'$ so that $w \neq q, q'$. ***Now*** $w \in X = \{a\} \cup a^\perp \cup \mathrm{co}(a^\perp)$. Note w cannot equal a. If $w \in a^\perp$, then

$w \in a^\perp \cap q^\perp$ so $w \in q'^\perp$ and $w \neq q$ so $w \in q'^\perp \backslash \{q\}$. If $w \in \mathrm{co}(a^\perp)$, then, since $w \neq p$, we have $w \in \mathrm{co}(a^\perp)\backslash\{p, q'\} = (p^\perp \cap \mathrm{co}(a^\perp)) \cup (q'^\perp \cap \mathrm{co}(a^\perp))$. But $w \notin p^\perp \cap \mathrm{co}(a^\perp)$ since otherwise $w \in q^\perp \cap p^\perp \cap \mathrm{co}(a^\perp) = \square$. Hence $w \in q'^\perp \cap \mathrm{co}(a^\perp)$ so $w \in q'^\perp$ and $w \neq q$.

(4) Let $w \in (q^\perp \cap \mathrm{co}(a^\perp))\backslash\{q'\}$. Then $w \in q^\perp \cap \mathrm{co}(a^\perp)$ and $w \neq q', q, p, a$. Now $w \in \mathrm{co}(a^\perp)\backslash\{p, q'\}$ and $w \notin p^\perp \cap \mathrm{co}(a^\perp)$ as above so $w \in q'^\perp \cap \mathrm{co}(d)$ and $w \neq q$.

(5) Note

$$\begin{aligned} q^\perp &= (q^\perp \cap a^\perp) \,\dot\cup\, (q^\perp \cap \mathrm{co}(a^\perp)) \\ &= (q^\perp \cap a^\perp) \,\dot\cup\, [(q^\perp \cap \mathrm{co}(a^\perp))\backslash\{q'\}] \,\dot\cup\, \{q'\} \\ &= (q'^\perp \cap a^\perp) \,\dot\cup\, [(q'^\perp \cap \mathrm{co}(d))\backslash(q)] \,\dot\cup\, \{q'\}. \end{aligned}$$

Now union on $\{q\}$.

(6) is clear from (5) and the canonical decomposition of X.

We next introduce the ***Shult counting functions.*** Let $(X, \perp)$ be a finite orthogonality space. Let $a \in X$. Define $m_a(x) = |x^\perp \cap a^\perp|$ and $s_a(x) = |x^\perp \cap \mathrm{co}(a^\perp)|$ for any x in X.

LEMMA 6.

(1) m_a *and* s_a: $X \to [0, |X|] \cap (N \cup \{0\})$,

(2) $m_a(x) = m_x(a)$,

(3) $m_a(a) = |a^\perp|$,

(4) $s_a(a) = 0$,

(5) *if* $x \in a^\perp$, then $|x^\perp| = 1 + m_a(x) + s_a(x)$,

(6) if $x \in \mathrm{co}(a^\perp)$, *then* $|x^\perp| = m_a(x) + s_a(x)$.

Proof. (1), (2), (3), and (4) are clear from definitions.

(5) Note, if $x \in a^\perp$, then $x^\perp = \{a\} \,\dot\cup\, (a^\perp \cap x^\perp) \,\dot\cup\, (\mathrm{co}(a^\perp) \cap x^\perp)$.

(6) Note, if $x \in \mathrm{co}(a^\perp)$, then $x^\perp = (a^\perp \cap x^\perp) \,\dot\cup\, (\mathrm{co}(a^\perp) \cap x^\perp)$.

A space $(X, \perp)$ is an NT ***space*** iff, for each $x \in X$, $x^\perp \neq \square$ and $\mathrm{co}(x^\perp) \neq \square$. If $x \in X$, the ***valence of*** x is $|x^\perp|$, and the ***co-valence of x*** is $|\mathrm{co}(x^\perp)|$. A space $(X, \perp)$ is ***regular*** of ***multiplicity*** α iff $|x^\perp| = \alpha$ for all x in X. It is called ***co-regular*** of ***co-multiplicity*** β iff $|\mathrm{co}(x^\perp)| = \beta$ for all x in X. A space is called ***point determining*** iff whenever $x^\perp = y^\perp$ we have $x = y$.

LEMMA 7. *A regular NT space with the property that $s_a(x) > 0$ whenever x is in $\mathrm{co}(a^\perp)$ is point determining.*

Proof. Let X be such a space and suppose the multiplicity is α. Suppose $a^{\perp} = b^{\perp}$ but a $\neq b$. First note $m_a = m_b$. Thus $\alpha = \ a^{\perp} \ = $ ***m,(a) =*** $m_b(a) =$ ***m,(b).*** Clearly ***b*** is not in a$^{\perp}$. Then since $b \neq a$, ***b*** is in $\mathrm{co}(a^{\perp})$ so $\alpha = \ b^{\perp} \ =$ ***m,(b)*** + ***s,(b)*** = α + ***s,(b)*** so ***s,(b)*** = 0, a contradiction.

The space $(X, \perp)$ has the ***triangle property*** iff every orthogonal pair of points can be completed to a fundamental triple. A ***Shult space*** is a finite orthogonality space which is regular and has the triangle property. E. Shult has characterized all Shult spaces [5]. Trivial examples of Shult spaces are the ***complete*** (or ***Boolean) space*** C_n on n points, $n \neq 2$, and the ***dispersed*** (or ***scattered) space*** D_n on n points. If a Shult space is also an ***NT*** space we shall call it a ***proper Shult space.*** A space has the ***co-triangle property*** iff every non-orthogonal pair of distinct points can be completed to a fundamental co-triple. A proper Shult space with the co-triangle property is called a ***symplectic orthogonulity space.***

From now on, assume $(X, \perp)$ is a symplectic orthogonality space of multiplicity α. Note that (3) of the next lemma is essentially the argument of Shult in [4].

LEMMA 8.

(1) ***X is co-regular of co-multiplicity*** $\beta = \ X \ - \alpha - 1$,

(2) β ***is even,***

(3) if a $\in b^{\perp}$, ***then s,(b)*** = $\beta/2$,

(4) ***if a*** $\in \mathrm{co}(p^{\perp})$, ***then s,(p)*** = $(\beta - 2)/2$,

(5) if $x \in \mathrm{co}(a^{\perp})$, ***then s,(x)*** > 0. ***In particular X is point determining.***

Proof. (1) is clear.

(2) and (3) Let a $\perp$ ***b.*** Then the orthogonal pair (a, ***b)*** can be completed to a fundamental triple (a, ***b,*** c). By Lemma 2 ***{b)*** $\cup (b^{\perp}$ n $a^{\perp}) =$ {c} $\cup$ $(c^{\perp}$ n $a^{\perp})$ so $1 + \ b^{\perp}$ n $a^{\perp} \ = 1 + \ c^{\perp}$ n $a^{\perp}$ and so $b^{\perp}$ n $a^{\perp}$ I = $c^{\perp}$ n $a^{\perp}$ |. Thus ***m,(b)*** = ***m,(c).*** By Lemma 6, ***b*** $\in a^{\perp}$ and c $\in a^{\perp}$ imply $\alpha = \ b^{\perp} \ = 1 +$ ***m,(b)*** + ***s,(b)*** and $\alpha = \ a^{\perp} \ = 1 +$ ***m,(c)*** + s,(c). Thus ***s,(b)*** = s,(c). By Lemma 2, $\mathrm{co}(a^{\perp}) = (b^{\perp}$ n $\mathrm{co}(a^{\perp})) \cup (c^{\perp}$ n $\mathrm{co}(a^{\perp}))$ so $\mathrm{co}(a^{\perp})| = \ b^{\perp}$ n $\mathrm{co}(a^{\perp})| + \ c^{\perp}$ n $\mathrm{co}(a^{\perp})|$. Thus $\beta =$ ***s,(b)*** + s,(c) = $2s_a(b)$. We note in passing that ***m,(b)*** = $(3\alpha - \ X \ - 1)/2$.

(4) Let a, ***p*** be in X with a $\not\perp$ ***p*** and a $\neq$ ***p.*** Then (a, ***p)*** can be completed to a fundamental co-triple (a, ***p, q).*** By Lemma 4, $a^{\perp}$ n $p^{\perp} = a^{\perp}$ n $q^{\perp}$, ***so*** that ***m,(p)*** = ***m,(q).*** Also, by Lemma 4, we have $\mathrm{co}(a^{\perp}) \backslash \{p,$ ***q}*** = $(p^{\perp}$ n $\mathrm{co}(a^{\perp})) \cup (q^{\perp}$ n $\mathrm{co}(a^{\perp}))$ so

$$\mathrm{co}(a^{\perp})| - 2 = \ p^{\perp} \text{ n } \mathrm{co}(a^{\perp})| + \ q^{\perp} \text{ n } \mathrm{co}(a^{\perp})|.$$

That is, $\beta - 2$ = s,(p) + s,(q). By Lemma 6, since $p \in \mathrm{co}(a^{\perp})$ and $q \in \mathrm{co}(a^{\perp})$, $\alpha =$ $p^{\perp}$ I = $m_a(p) + s_a(p)$ and $\alpha =$ I $q^{\perp}$ I = $m_a(q) + s_a(q)$. Thus **s,(p) = s,(q).** Hence β 2 = **s,(p) + s,(q)** = $2s_a(p)$.

(5) Suppose x $\in \mathrm{co}(a^{\perp})$ but s,(x) = 0. By the ***NT*** property, there exist $b \in a^{\perp}$. By the triangle property, there exists c $\in a^{\perp}$ such that (a, ***b, c)*** is a fundamental triple. Now s,(x) = 0 means x is orthogonal to nothing in $\mathrm{co}(a^{\perp})$. But also x $\not\perp$ a. However, x is orthogonal to α points of X. Hence, x is orthogonal to everything in $a^{\perp}$. In particular, x $\perp$ ***b,*** x $\perp$ c but x $\not\perp$ a, contradicting that (a, ***b,*** c) is a fundamental triple.

LEMMA 9 (Uniqueness Lemma). ***In a symplectic orthogonality space, any orthogonal pair can be completed uniquely to a fundamental triple and any two fundamental triples have at most one point in common. Any distinct pair of non-orthogonal elements can be completed uniquely to a fundamental co-triple and any two fundamental co-triples have at most one point in common.***

Proof. First let (a, ***b,*** c) and (a, ***b,*** c') be two fundamental triples with c $\neq$ c'. By Lemma 3(3), c $\perp$ c'. By Lemma 3(6), we have $\mathrm{co}(c^{\perp}) = \mathrm{co}(c'^{\perp})$ so for any x in X we have $x^{\perp} \cap \mathrm{co}(c^{\perp}) = x^{\perp} \cap \mathrm{co}(c'^{\perp})$. Thus s,(x) = $s_{c'}(x)$. In particular, 0 = s,(c) = $s_{c'}(c) = \beta/2$. Hence $\beta = 0$, contradicting the ***NT*** property.

Next let ***(a, p, q)*** and (a, ***p, q')*** be two fundamental co-triples with $q \neq$ ***q'.*** By Lemma 5(2), $q \perp$ ***q'.*** By Lemma 5(6), $\mathrm{co}(q^{\perp}) = \mathrm{co}(q'^{\perp})$. Thus s,(x) = $s_{q'}(x)$ for all x. Thus 0 = **s,(q)** = $s_{q'}(q) = \beta/2$. Hence $\beta = 0$, again a contradiction.

If a and ***b*** are in X with ***a*** $\perp$ ***b,*** the unique c in X such that ***(a, b, c)*** is a fundamental triple is denoted $e(a,$ ***b).*** If ***a*** and ***p*** are in X with ***a*** $\not\perp$ ***p*** and a $\neq$ ***p,*** the unique q such that ***(a, p, q)*** is a fundamental co-triple is denoted $f(a,$ ***p).*** The next is an immediate corollary to the Uniqueness Lemma.

LEMMA 11. *Let $a, b \in X$ with $a \perp b$ and $p \in X$ with $a \not\perp p$ and $a \neq p$.* ***Then***

(1) $e(a, b) = e(b, a)$,

(2) $f(a, p) = f(p, a)$,

(3) $e(a, e(a, b)) = b$,

(4) $f(a, f(a, p)) = p$.

Now let $\vec{0}$ denote some element not in X. Let $E = X \cup \{\vec{0}\}$. Define an addition operation on E as follows:

'(1) $\vec{0} + \vec{0} = \vec{0}$;

(2) if $x \in X$, $\vec{0} + x = x + \vec{0} = x$;

(3) let $x, y \in X$;

(i) if $x = y$, define $x + y = \vec{0}$,

(ii) if $x \perp y$, define $x + y = e(x, y)$,

(iii) if $x \not\perp y$, $x \neq y$, define $x + y = f(x, y)$.

Also define an action of the Galois field GF(2) on E by $0x = \vec{0}$, and $1x = x$ for all x in E. Then all the vector space axioms for E are clear with the exception of the associative law, which is considered next.

LEMMA 12 (Lemma of the Odd Triples). ***Let $\{x, y, z\}$, $\{x, t, w\}$, $\{w, y, p\}$, and $\{z, t, q\}$ be odd triples. Then $p^{\perp} = q^{\perp}$ and hence by the point determining property, $p = q$.***

Proof. Let $a \in q^{\perp}$. Show $a \in p^{\perp}$. By oddness, it suffices to show $a \perp y$ and $a \perp w$ or to show $a \not\perp y$ and $a \not\perp w$. If $a \perp t$, then necessarily $a \perp z$. Thus either $a \perp x$ and $a \perp y$ or $a \not\perp x$ and $a \not\perp y$. In case $a \perp x$ and $a \perp t$, then $a \perp w$. Also $a \perp y$ so $a \perp p$. In case $a \not\perp x$, then $a \perp w$ since $a \perp t$. Also $a \not\perp y$ so $a \perp p$. On the other hand, if $a \not\perp t$, then necessarily $a \not\perp z$. Thus either $a \perp x$ or $a \perp y$ but not both. If $a \perp x$, then $a \not\perp t$ implies $a \not\perp w$. Also $a \not\perp y$ so $a \perp p$. If $a \perp y$, then $a \not\perp x$ so $a \perp w$ since $a \not\perp t$. Thus $a \perp p$. In all cases, $a \in p^{\perp}$. Thus $q^{\perp} \subseteq p^{\perp}$. Similarly $p^{\perp} \subseteq q^{\perp}$.

With the help of the above lemma, we can now argue the associative law.

LEMMA ***13.*** ***$t + (x + y) = (t + x) + y$.***

Proof. First, if $t = x = y$, there is nothing to prove. Next, if any one of x, y, t is $\vec{0}$, the result is clear so we may assume all are non-zero.

Case 1. Two ***of*** $x, y,$ ***t are equal.*** The possibilities are: (i) $t = x$, y different, (ii) $t = y$, x different, and (iii) $x = y$, t different. In Case (i), $(t + x) + y = y$. If $x \perp y$, then $t + (x + y) = x + (x + y) = e(x, e(x, y)) = y$ by Lemma 11. If $x \not\perp y$, then $x + (x + y) = f(x, f(x, y)) = y$ by Lemma 11. In Case (ii), we need to show $y + (x + y) = (y + x) + y$. But this is clear by the commutativity of addition. In Case (iii), $t + (x + y) = t + (y + y) = t$. If $y \perp t$, then $(t + x) + y = (t + y) + y = e(e(t, y), y) = t$. If $y \not\perp t$, then $(t + x) + y = f(f(t, y), y) = t$.

Case 2. All three of t, x, y are distinct. There are eight possibilities: (i) $x \perp t, t \perp y, y \perp x$, (ii) $x \not\perp t, t \not\perp y, y \not\perp x$, (iii) $x \perp t, t \not\perp y, y \not\perp x$, (iv) $x \not\perp t, t \perp y, y \perp x$, (v) $x \perp t, t \perp y, y \not\perp x$, (vi) $x \not\perp t, t \perp y, y \perp x$, (vii) $x \perp t, t \not\perp y, y \not\perp x$, and (viii) $x \not\perp t, t \not\perp y, y \perp x$. Note in Case (i) we have $t \perp e(x, y)$ and $y \perp e(t, x)$. Thus $\{x, y, e(x, y)\}$, $\{x, t, e(x, t)\}$, $\{e(x, t), y, e(e(t, x), y)\}$, and $\{e(x, y), t, e(t, e(x, y))\}$ are odd triples. By the Lemma of the Odd Triples, $e(e(t, x), y) = e(t, e(x, y))$; that is, $(t + x) + y = t + (x + y)$. In Case (ii), we have $t \perp f(x, y)$ and $y \perp f(t, x)$. Thus $\{x, y, f(x, y)\}$, $\{x, t, f(t, x)\}$, $\{f(x, y), t, e(f(x, y), t)\}$, and $\{f(t, x), y, e(f(t, x), y)\}$ are odd triples. Again by Lemma 12 $e(f(x, y), t) = e(f(t, x), y)$; that is, $t + (x + y) = (t + x) + y$. The remaining cases are similar reductions to Lemma 12.

Lemma 14. ***E is a vector space over* GF(2) *whose one-dimensional subspaces are in one-one correspondence with the points of X.***

Next, we introduce a symplectic bilinear form on E. Define $\Phi: E \times E \to \mathrm{GF}(2)$ by $\Phi(\vec{0}, x) = \Phi(x, \vec{0}) = 0$ for all x in E, $\Phi(x, x) = 0$ for all x in E and for any two distinct x, y in $E \backslash \{\vec{0}\}$, $\Phi(x, y) = 0$ if $x \perp y$ and 1 if $x \not\perp y$.

Lemma 15. ***Φ is a non-degenerate symplectic bilinear form on E.***

Proof. Clearly, $\Phi(x, y) = \Phi(y, x)$. We claim $\Phi(x, y + z) = \Phi(x, y) + \Phi(x, z)$. If $x = \vec{0}$, the claim is clear. If at least one of y or z is $\vec{0}$, it is clear. Assume x, y, z all non-zero.

***Case* 1.** $y \perp z$. Then $y + z = e(y, z)$. If $x \perp y$ and $x \perp z$, then $x \perp e(y, z)$ or $x = e(y, z)$. Thus $\Phi(x, y + z) = \Phi(x, e(y, z)) = 0$. But also $\Phi(x, y) = 0$ and $\Phi(x, z) = 0$. If x is orthogonal to only one of y or z, say y, then $x \not\perp z$ and $x \not\perp e(y, z)$. Then $\Phi(x, y + z) = \Phi(x, e(y, z)) = 1$. But also $\Phi(x, y) = 0$ and $\Phi(x, z) = 1$. Finally, if $x \not\perp y$ and $x \not\perp z$, then $x \perp e(y, z)$. Thus $\Phi(x, y + z) = \Phi(x, e(y, z)) = 0$. But also $\Phi(x, y) = 1$ and $\Phi(x, z) = 1$.

***Case* 2.** $y \not\perp z$. Then $y + z = f(y, z)$ if $y \neq z$ and is 0 if $y = z$. For $y = z$, $\Phi(x, y + z) = \Phi(x, \vec{0}) = 0$. Also $\Phi(x, y) = \Phi(x, z)$ so $\Phi(x, y) + \Phi(x, z) = 2\Phi(x, y) = 0$. Now suppose $y \neq z$. If $x \perp y$ and $x \perp z$, then $x \perp f(y, z)$. Thus $\Phi(x, y + z) = \Phi(x, f(y, z)) = 0$. But also $\Phi(x, y) = \Phi(x, z) = 0$. If x is orthogonal to only one of y or z, say y, then $x \not\perp f(y, z)$. Thus $\Phi(x, y + z) = \Phi(x, f(y, z)) = 1$. But also $\Phi(x, y) = 0$ and $\Phi(x, z) = 1$. Finally, if $x \not\perp y$ and $x \not\perp z$, then $x \perp f(y, z)$ or $x = f(y, z)$. Thus $\Phi(x, y + z) = \Phi(x, f(y, z)) = 0$. But also, $\Phi(x, y) = 1$

and $\Phi(x, z) = 1$. By the *NT* property, no non-zero element of E is orthogonal to all elements of E so Φ is non-degenerate.

We now come to the main result.

THEOREM. *Let $(X, \perp)$ be a symplectic orthogonality space. Then there is a symplectic quadratic space* $(\mathrm{GF}(2), E, \Phi)$ *such that $(X, \perp)$ is isomorphic to* $(\mathrm{lin}(E), \perp(\Phi))$.

Proof. Given such an $(X, \perp)$, construct E and Φ as above. Then, since the line space of E is essentially the same as $(E\backslash\{\vec{0}\}$, *I(@)),* the identity map between $E\backslash\{\vec{0}\}$ and X establishes the isomorphism.

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